Uncommon Sense

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Outline

What is Mathematics?

What is mathematics, and is it important?

Introduction

Many a true word spoken in jest

Bluff your way in Maths, Robert Ainsley

"There are a lot of books about mathematics — usually very long ones with thousands of pages of small print without pictures, full of strings of odd-looking, apparently meaningless characters — like any university maths faculty. We can, however, classify the function of mathematics quite simply. Mathematics consists essentially of:

1. *proving the obvious;*
2. *proving the not so obvious; and*
3. *proving the obviously untrue."*

Proving the obvious.

Birthdays

Question?

What is the smallest number of people that one must have in one room to be sure that two of them share the same birthday?

Answer?

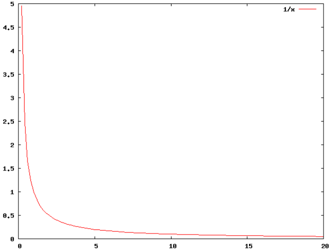
Many will say the answer is 366, but more pedantic observers will say that it is 367, one more than the number of days in a leap year.

The pigeon-hole principle

If $n > m$ pigeons are put into m pigeonholes, there's a hole with more than one pigeon.

Proving the obvious.

Infinite Areas?

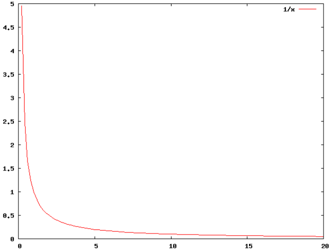


Obvious?

It's "obvious" that the area under the graph $y = 1/x$ from $x = 1$ to $x = \infty$ in infinite, since the graph never touches the x-axis.

Proving the obvious.

Infinite Areas?



And it is infinite

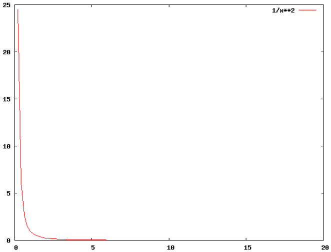
$$\int_1^\infty \frac{1}{x} dx = \lim_{k \to \infty} \int_1^k \frac{1}{x} dx$$

$$= \lim_{k \to \infty} [\log_e x]_1^k$$

$$= \lim_{k \to \infty} (\log k - 0) = \infty$$

Proving the not so obvious.

Infinite Areas?

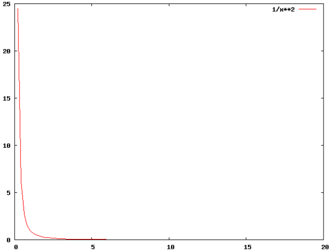


Is it obvious?

It's also "obvious" that the area under the graph $y = 1/x^2$ from $x = 1$ to $x = \infty$ in infinite, since the graph never touches the x-axis. Although as the figure shows it gets incredibly close... but that shouldn't matter, should it?

Proving the not so obvious.

Infinite Areas?



It is not true!

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{k\to\infty}\int_1^k \frac{1}{x^2}\,dx$$

$$= \lim_{k\to\infty}\left[-\frac{1}{x}\right]_1^k$$

$$= \lim_{k\to\infty}\left(-\frac{1}{k} - -\frac{1}{1}\right) = 1$$

Proving the not so obvious.

Infinite Areas?

Results from the real line

This result stems from, and has obvious analogs in the analysis of the real number line. For example, it is "obvious" that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$$

and not only is it obvious, it is true! It is less obvious that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

has no such answer.

Proving the not so obvious.

Infinite Areas?

OK, but so what?

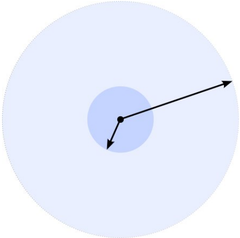
These results are interesting, and tell us that something important happens when we start adding up infinitely many things that are becoming infinitely small. There is a delicate balancing act here. So much for the pure mathematics.

Does it impact on the real world?

In fact, the ability to perform such integrals and get answers that are not infinite have important and interesting consequences in the mathematics that models the real world.

Proving the not so obvious.

Infinite Areas?



Radiating Force

In classical physics forces are propogated by fields that (in general) spread out equally in all directions. If one considers the force “radiating” out from the source, then this force is spread increasingly thinly as the radius r from the source increases. Consider the force spread out at a specific radius r, then it is distributed over a surface area $4\pi r^2$.

Proving the not so obvious.

Infinite Areas?

The Inverse Square Law

As the original force is spread out over this sphere it follows that for these forces:

$$F = \frac{k}{r^2}$$

where k is a (sometimes controversial) constant.

Examples

Many forces follow the inverse square law, but the most common secondary school examples are Newton's universal law of gravitation and Coulomb's electrostatic law that both follow this pattern.

Escape from Earth, or anything else for that matter.

Work Done is...

Consider the energy necessary for an object to be launched into space. We can use the more complete version of $E = Fs$, where E is the work done, F is the force and s the distance travelled, since the force here is not a constant, we must use calculus:

$$E = \int_a^b F \, ds$$

gives the work done by a force in moving an object from $s = a$ to $s = b$.

Proving the not so obvious.

Escape from Earth, or anything else for that matter.

The mathematics

$$E = \int_{r_E}^{\infty} Fs\,ds = k\int_{r_E}^{\infty} \frac{1}{s^2}\,ds$$

Now because this is essentially $y = 1/x^2$, this integral is actually finite.

$$E = \frac{k}{R_E} = \frac{GM_E m}{R_E}$$

Why it's important

If this were not so, it would be impossible for any object to be launched into space, or for any electron to be knocked out of orbit of an atom.

Proving the not so obvious.

Escape from Earth, or anything else for that matter.

This can be used to calculate the escape velocity for Earth, quite easily.

By the way

$$\frac{1}{2}mv^2 = \frac{GM_E m}{R_E} \Rightarrow v = \sqrt{\frac{2GM_E}{R_E}}$$

Schwarzchild Radius of a Black Hole R_B

Or to find the radius of a black hole in a similar way! Surprisingly, for a non rotating black hole, this is correct!

$$\frac{1}{2}mc^2 = \frac{GM_B m}{R_B} \Rightarrow R_B = \frac{2GM_B}{c^2}$$

Proving the not so obvious.

Pigeon holing people again

Alexander Bogomolny

"At any given time in New York there live at least two people with the same number of hairs."

Alexander Bogomolny

" I ran experiments with members of my family. My teenage son secured himself the highest marks sporting, in my estimate, about 900 hairs per square inch. Even assuming a pathological case of a 6 feet (two-sided) fellow 50 inch across, covered with hair head, neck, shoulders and so on down to the toes, ...

Proving the not so obvious.

Pigeon holing people again

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the fellow would have somewhere in the vicinity of 7,000,000 hairs which is probably a very gross over-estimate to start with. The Hammond's World Atlas I purchased some 15 years ago, estimates the population of the New York City between 7,500,000 and 9,000,000. The assertion therefore follows from the pigeonhole principle."

Proving the obviously untrue.

However, it is in the latter of Ainsley's categorisations that mathematics is often at its most interesting - in proving the obviously untrue.
These are the cautionary cases that explain the caution of mathematics, after all. Sometimes the obvious is wrong!

Proving the obviously untrue.

Roping the world

A long piece of rope

Imagine one has a rope long enough to straddle the whole way around the world (say on the equator).
Now, plant sticks to sit 1 m up from the surface of the earth.
How much extra rope to we need to go round the Earth 1 m above its surface?

Proving the obviously untrue.

Roping the world

A long piece of rope

- Of course, it's a clue that the radius of the Earth is not given. Let's call it R_E.
- The original rope has length $C_E = 2\pi R_E$,
- The new length is $C_N = 2\pi(R_E + 1)$.

So

$$C_N = 2\pi R_E + 2\pi \Rightarrow C_N = C_E + 2\pi$$

so only about another 6.28 m.

Proving the obviously untrue.

1=2

Prove 1=2

Let $a = b$.

$$\Rightarrow a^2 = ab \Rightarrow a^2 + a^2 = a^2 + ab \Rightarrow 2a^2 = a^2 + ab$$

$$\Rightarrow 2a^2 - 2ab = a^2 + ab - 2ab \Rightarrow 2a^2 - 2ab = a^2 - ab$$

$$\Rightarrow 2(a^2 - ab) = 1(a^2 - ab)$$

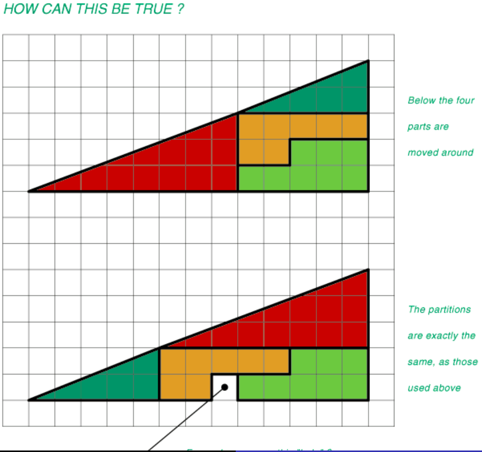
Now cancel $(a^2 - ab)$ both sides and we obtain

$$1 = 2$$

Proving the obviously untrue.

Back to areas...

Here is an example that arrived on my desk one day.



Proving the obviously untrue

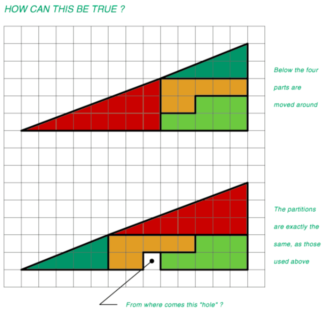
Back to areas...

Warning!

If you don't want to see the answer, look away now!

Proving the obviously untrue.

Back to areas...



Solution

- Look at the gradients of the red and green triangle upslope. If they were similar to the large triangle (which they should be) then they would be the same.
- The green triangle gradient is $2/5$
- The red triangle gradient is $3/8$
- The big triangle isn't actually a triangle.

Mobius strips

Take a strip of paper, give one end a half (180 degrees) twist and tape them together.

- how many sides does it have?
- how many edges?
- what happens when you cut it in half?
- or cut a third off the end?
- what use is it?

Proving the obviously untrue

The Spider and the Fly

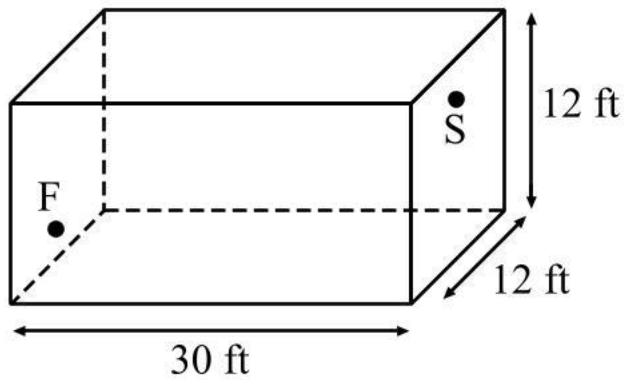
This fascinating example of the obviously untrue is accessible to almost all students, relying as it does on only the most elementary of mathematics.

We consider a room of dimensions 12 feet tall by 12 feet across by 30 feet long (since this is an old puzzle, or a very big room if we use metres).

The room has two occupants.

Proving the obviously untrue

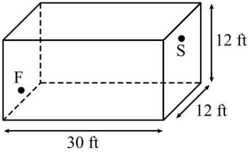
The Spider and the Fly



Spider centered on one wall, one foot from the ceiling.
Fly centered on opposite wall, one foot from the floor.

Proving the obviously untrue

The Spider and the Fly



Spider centered on one wall, one foot from the ceiling.
Fly centered on opposite wall, one foot from the floor.

Assumptions

1. the spider is very hungry, too hungry to spin silk;
2. therefore the spider stays on the walls at all times;
3. the spider has a degree in mathematics, or cognate subject.

Question?

What is the shortest distance the spider must travel to collect her prize? The *obvious* answer is 42, but it is wrong...

Proving the obviously untrue

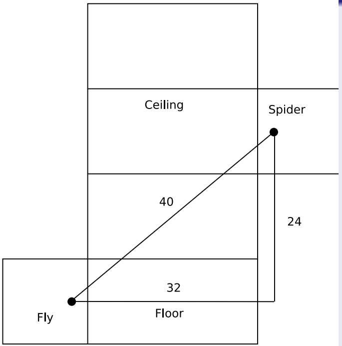
The Spider and the Fly

Warning!

If you don't want to see the answer, look away now!

Proving the obviously untrue

The Spider and the Fly



Solution

- As you can see, the correct answer travels over 5 out of the 6 walls!
- There are other possible unfoldings of course, some of which give the 42 answer we say before.
- Some students have complained this makes the whole thing a trick, since one can't unfold a real room the whole thing is impossible.

Proving the obviously untrue

The Linking Rings

Most of us will have seen the linking rings illusion; where a magician will link and unlink solid metal rings. This is clearly obviously untrue, or is it?
We shall consider another way of looking at this age old illusion.

The value of the obviously untrue

In fact these proofs are often the most informative. They give us deep insight into our assumptions and allow us to see the true structure of things.

Mathematics is about patterns, where they work and were they break. Patterns can be found in number, music, art, language, science, engineering . . .

Summary

- The obvious is often false
- The obviously false is often true
- The examples that demonstrate these two crossovers in perception are often the most useful in demonstrating the real nature of things.